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Infinite-dimensional time vectors as background building blocks of a space–time frame structure

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The convenient properties of cumulative probability density functions are used to describe how sequences of infinite-dimensional time vectors can be generated from them. It is also discussed how such a description can be connected with space–time frames, while conserving a coherent connection related to the usual experimental scalar measures of time.

KEY WORDS: space-time frames, time vectors, vector semispaces, inward matrix products, probability density functions, minkowski norms

1. Introduction

In a previous paper [1], the classical point of view of time as a scalar parameter was substituted by an N-dimensional *time vector* structure, following somehow the spirit of an old idea borrowed from early quantum mechanical field theory [2]. Classical time positive definite form demands, however, that these time vectors, substituting scalar time, behave in some sense as the scalar usual parameter and, in this manner; time vectors have been constructed in such a way as belonging to a *vector semispace* [3]. A vector semispace is a vector space defined over the positive definite real field, where the additive group is taken instead as an *additive semigroup* [4], so zero and reciprocal element vectors are not present in semispaces. Also, within this general framework, *time reversal* structure could just be considered as a symmetric situation of the semispace time vectors and, as such, there is no need to be further discussed, as it will have the same structure present in usual time vectors positive definite directions, but with a sign reversed reciprocal *signature* [9].

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If it is assumed that time vectors belong to some semispace, every one of them can be generated by a vector pertaining to the space where the active semispace is contained. That is, just employing an *inward matrix product* structure [5], one can simply write the generation of time vector semispace as

$$\forall \mathbf{t} \in U(\mathbf{R}^{+}) \subset V(\mathbf{C}) \rightarrow \exists \mathbf{z} \in V(\mathbf{C}) : (\mathbf{z}^{*})\mathbf{z} = \mathbf{t}, \tag{1}$$

this description just means that

$$\mathbf{t} = \{t_I\} \wedge \mathbf{z} = \{z_I\} \rightarrow \forall I : t_I = |z_I|^2 . \tag{2}$$

This straightforward construction can be described in short by a *generating symbol* [3]

$$R(\mathbf{z} \rightarrow \mathbf{t}) \equiv \left\{ \mathbf{t} = (\mathbf{z}^{*})\mathbf{z} = |\mathbf{z}|^2 \right\} . \tag{3}$$

The generating symbol implicit algorithm, depicted in the previous definitions, encompass any vector space and its attached semispace, including Hilbert space constructs. The reader can recognize the generating symbol as the procedure of quantum mechanical construction of the density function: $\rho(\mathbf{r})$, associated to a given state wave function, $\Psi(\mathbf{r})$. As it can be written, according to the generating symbol definition adapted to functional spaces:

$$R(\Psi \rightarrow \rho) \equiv \left\{ \rho = |\Psi|^2 \right\} \tag{4}$$

with the inward product squared module of equation (3) transformed into a simple product squared module.

Accordingly, to every time vector semispace there can be attached a generating vector space.

2. Extended functional time vectors

The success of substituting scalar time with time vector structure and the possible definition of density functions in general Hilbert functional spaces by means of the generating symbol (4), may suggest the possibility to employ density functions as time vector components or simply as time vectors by themselves. Indeed, probability density functions possess the adequate properties to behave as time vectors, although within an infinite dimensional framework. In this sense, time may be thought as possessing the structure belonging to a positive definite function. Nothing has to be *a priori* supposed about the nature of the variables associated to the time density functions. Thus, infinite dimensional time vector forms can be considered as the result of a non-linear transformation of some variable set, which can be attached in turn to a complex system description; in a parallel way as quantum mechanical density functions are considered.

Time measures shall be, then, associated to some manipulation performed over time density, providing a positive definite scalar result. Such a possibility is naturally included into the time density definition, as for instance integrals over positive definite functions, even if weighted by positive definite operators, provide the basis for a measure evaluation. As an application example of this possibility, it is worth quoting the use of quantum mechanical density functions for the definition of *quantum similarity measures* [6].

Now, in order to define a possible practical framework for this time density picture, there should be taken into account several details concerning the possible construction of vector spaces depending of such time structure.

3. Time density vectors and position space components

For this purpose, suppose a position vector $\mathbf{x}$ in an N-dimensional parameterized space, which will be used as a space–time frame. Furthermore, suppose that the vector $\mathbf{x}$ components depend on different time elements taken provisionally as parameters, as it was previously defined [1]:

$$\forall \mathbf{x}(\mathbf{t}) \in V_N(\mathbf{R}) \wedge \mathbf{t} \in U_N\left(\mathbf{R}^+\right) \rightarrow \mathbf{x}(\mathbf{t}) = \{x_I\,(t_I)\}\,. \tag{5}$$

From this definition as above stated, there is only additionally needed in order to have a starting infinite-dimensional time structure possibility, to consider every time vector component as a time density function $\tau(\mathbf{r})$ of some variable set $\{\mathbf{r}\}$, which can be attached, as already commented, to some complex system description. It can be written, for example,

$$\forall \mathbf{t} = \{t_I\} \rightarrow t_I \equiv \tau_I\,(\mathbf{r}_I) \in U_\infty\left(\mathbf{R}^+\right), \tag{6}$$

by assigning to each vector component a different time density function. A more classical situation, while keeping this time density structure at a minimal pace, may consist in considering all time vector density components equal to a unique density function: $\rho(\mathbf{r})$, that is,

$$\forall I : \tau_I\,(\mathbf{r}_I) \equiv \rho(\mathbf{r}) \in U_\infty\left(\mathbf{R}^+\right) \rightarrow \mathbf{x}(\mathbf{t}) \equiv \{x_I(\rho(\mathbf{r}))\} \vee \mathbf{t} = \rho(\mathbf{r})\mathbf{1}, \tag{7}$$

where the symbol $\mathbf{1}$ is taken as the *unity* vector, which acts as the neutral element in inward matrix products. It is used here to express an N-dimensional array whose components are the scalar 1.

Still, such time density picture may not be completely satisfactory from the point of view of our perception about the nature of time progression, but is a device defined just to obtain a positive definite time vector structure. And so, there may be needed some additional detailed analysis around the possibility that time density reflects such a property attached to the progressive nature of time measures too.

4. Time density and time progression measures

As it has been commented before, time density functions, associated to a mathematical structure such as probability density functions, first of all should be able to be defined within a set of generating functions belonging to some Hilbert space. Then, due to this framework, any generated time density $\rho(\mathbf{r})$, corresponding to the generating symbol $R(\Psi \rightarrow \rho)$ can be also normalized in the Minkowski sense, as

$$\langle \rho \rangle = \int_D \rho(\mathbf{r})\, d\mathbf{r} = 1, \tag{8}$$

because one can use the usual Euclidean norm of the generating Hilbert space function:

$$\langle \rho \rangle = \int_D |\Psi(\mathbf{r})|^2\, d\mathbf{r} = \langle \Psi / \Psi \rangle = 1, \tag{9}$$

as it is customarily done in basic well-known quantum mechanical theory [7]. In both expressions (8) and (9), D corresponds to an appropriate normalization domain of the underlying system variable set $\mathbf{r}$.

Moreover, as it is usually done in theoretical statistics [8], a *cumulative* probability density can be also defined, using as basic density any normalized density function, $\rho(\mathbf{r})$, like in the present example:

$$\chi(\mathbf{s}) = \int_{S \subseteq D} \rho(\mathbf{r})\, d\mathbf{r} \rightarrow \chi(\mathbf{s}) \in [0, 1], \tag{10}$$

where S is an appropriate subdomain of the norm defining domain D; while the new variable vector set $\{\mathbf{s}\}$ can be built up as corresponding to the upper limit of the subdomain S, whenever the lower limit is kept constant over the integration sequence.

Constructed in this way, the positive definite function $\chi(\mathbf{s})$ can be easily associated to a sequence of measures, yielding in turn a sequence of increasing positive definite scalar values, lying in the closed interval [0,1], whenever a sequence of increasing subdomain volumes is employed, that is,

$$S_0 \subset S_1 \subset \cdots \subset S_N \rightarrow \chi(\mathbf{s}_0) < \chi(\mathbf{s}_1) < \cdots < \chi(\mathbf{s}_N). \tag{11}$$

Thus, the sequence of cumulative probability density measures $\{\chi(\mathbf{s}_I)\}$, conveniently attached to an origin value and scaled, can be put into a one to one correspondence to an increasing sequence of scalar time measures, which can be written as

$$t_0 < t_1 < \cdots < t_N. \tag{12}$$

This simple property permits to state a fundamental property of infinite dimensional time structures, as follows. Given some time density function $\rho(\mathbf{r})$, Minkowski normalized in the sense of equation (8) over some system variable domain D, then any positive definite integral function $\chi(\mathbf{s})$ value sequence, as defined in equation (10), representing a cumulative probability within some subdomain $S \subseteq D$ sequence, can be attached to a sequence of *time measures*.

The proposed time vector structure in parameterized position spaces of arbitrary dimension can be, at the light of the last discussion, redefined by means of the cumulative probability functions as

$$\forall \mathbf{t} = \{t_I\} \rightarrow t_I \equiv \chi_I(\mathbf{s}_I) \in U_\infty\left(\mathbf{R}^+\right) \wedge \chi_I(\mathbf{s}_I) = \int_{S_I \subseteq D_I} \tau_I(\mathbf{r}_I)\,\mathrm{d}\mathbf{r}_I, \qquad (13)$$

in such a way that the time vector argument $\mathbf{t}$, acting as a parameter set within the position vector $\mathbf{x}(\mathbf{t})$, has a proper behavior; as each of its elements can be associated to an ordered set of time measures, which can be considered, when holding well-defined values, as an element of some N-dimensional time vector semispace.

5. Time measures in *N*-dimensional time vectors

Although the measure of time in the proposed infinite dimensional representation has been previously studied and employed to construct appropriate N-dimensional arrays of time sequences, to be further included as parameters in the space position N-dimensional vectors of a space–time frame, it has not yet been discussed at all how this time structure can be connected to the usual scalar time measurements. It is interesting to study this question, as it was not done in the preceding study [1], where just the most immediate consequences of a non scalar time structure were mostly analyzed.

Due to the definition of the N-dimensional time vector as given in equation, based on the cumulative probability density functions, picking up two time vectors in a possible sequence of these arrays, $\mathbf{t}_I$ and $\mathbf{t}_J$ say, then the following sums of elements can be easily obtained, which are coincident with the time vectors Minkowski norms:

$$\theta_I = \langle \mathbf{t}_I \rangle \rightarrow S(\theta_I) \wedge \theta_J = \langle \mathbf{t}_J \rangle \rightarrow S(\theta_J),$$

where the symbols $S(\theta) \subseteq U_N\left(\mathbf{R}^+\right)$ stand for the collection of vectors of the semispace $U_N\left(\mathbf{R}^+\right)$, possessing the elements sum equal to the value θ, and constituting in this way a θ-*shell* [9].

With this shell definition in mind, let us suppose that the pair of vector Minkowski norms are related by the inequality: $\theta_I < \theta_J$. Then, at least, there exists an element of the vector $\mathbf{t}_J$, which will be greater than an element corresponding to the array $\mathbf{t}_I$. In this situation one can say that the vector $\mathbf{t}_I$ *precedes*

$\mathbf{t}_J$ and this situation can be depicted as: $\mathbf{t}_I < \mathbf{t}_J$. Thus, a sequence of time arrays: $\{\mathbf{t}_P\}$, acting as parameters in some position vector attached sequence, possessing a sequence of θ-shell values: $\Theta = \{\theta_P\}$, ordered in the preceding sense, as stated above, can be associated into a one to one correspondence, to an ordered scalar time sequence: $\Xi = \{\xi_P\}$, say, by shifting and scaling the Minkowski norm values:Θ.

Proceeding in this way, however, one can see how several essentially independent time vectors can produce an equivalent Minkowski norm. This may be due to the fact that scalar time measures, as defined by the time vector Minkowski norms, may be degenerate when applied to some strictly different vectors. However, in this case one can consider then the appearance of a scalar time *simultaneity* situation, associated to essentially different and independent time vector descriptors.

It can be concluded, in consequence, that Minkowski norms of time vector ordered sequences, acting as parameters into N-dimensional space–time frames, can be used as scalar time measures, providing information on how time evolves at the usual experimental observable scale for some points located within space–time frames. Objects and object sets can be described in space–time in this way, without apparently contradict the usual perception and measure of scalar time.

6. Conclusions

Time structure can be defined possessing an infinite-dimensional background as provided by cumulative probability density functions. Semispace time structure can be set with arbitrary dimensionality and considered as generated throughout inward matrix products of vectors, belonging to well-defined vector spaces where the time semispace belongs, in a parallel way as quantum density functions are generated. Sets of such positive definite probability density functions can be used as arguments, providing the time parameter values, within the space vectors constructed in N-dimensional space–time frames. Scalar time measures can be easily obtained in order to establish the connection between this multidimensional time structure and the usual observed progression of experimental scalar time measures.

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